Indian Statistical Institute, Bangalore
M. Math.
Second Year, First Semester
Operator Theory

Semestral Examination — Date : Dec. 8, 2010
Maximum marks: 100 — Time: 3 hours

In the following the field for vector spaces and algebras is taken to be the field of complex numbers and σ denotes spectrum.

(1) Suppose U is a unitary matrix then show that there exists a self-adjoint matrix H such that $U = e^{iH}$. [10]

(2) Show that the set of invertible elements in a unital Banach algebra is an open set. [15]

(3) Let $\mathcal{C}$ be a unital Banach algebra and $x \in \mathcal{C}$. If $\mathcal{C}$ is commutative show that the following are equivalent:
(i) x is not invertible;
(ii) There exists a maximal (two sided) ideal I of $\mathcal{C}$ such that $x \in I$.
Give an example to show that this result need not hold if $\mathcal{C}$ is not commutative.
[20]

(4) Let D be a $n \times n$ positive definite matrix of with trace $(D) = 1$. Let $M_n(\mathbb{C})$ be C^*-algebra of all $n \times n$ complex matrices. Show that ψ defined by $\psi(X) =$ trace (DX) is a state on $M_n(\mathbb{C})$. [15]

(5) Let ϕ be a state on a unital C^*-algebra $\mathcal{A}$. Show that $\mathcal{N} = \{b : \phi(b^*b) = 0\}$ is a closed left ideal of $\mathcal{A}$. [15]

(6) Let ϕ be a bounded linear functional on a unital C^*-algebra. Show that ϕ is positive if and only if $\|\phi\| = \phi(1)$. [20]

(7) Let $M_3(\mathbb{C})$ be the algebra all 3×3 complex matrices. Show that there is no non-trivial $*$-homomorphism from $M_3(\mathbb{C})$ to the algebra of complex numbers.
[10]